ASPECTS OF THE DYNAMICS OF A NONLINEAR VISCOELASTIC FLUID

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UDC 532.501.33:532.135

Boundary-value problems for systems of equations describing steady flows of a nonlinear viscoelastic fluid are reduced to variational problems.

§1. Nonlinear Viscoelastic Fluids

The stressed state in a nonlinear viscoelastic fluid is determined by the strain history. In a reasonably general situation the stress tensor σ at time t in an incompressible nonlinear viscoelastic fluid that is isotropic in the undeformed state is determined by the expression [1]

$$\sigma(t) = -pI + \overset{\infty}{\underset{s=0}{F}}[E(t-s)]. \tag{1}$$

Here p is the hydrostatic pressure; I is the unit tensor; and F is an isotropic symmetric nonlinear tensor functional of the tensor function E(t − s), which is a measure of the strain from the time t to the time t − s. This function has the form [2]

$$E_{ij}(t-s) = \frac{\partial x_i}{\partial \chi_a}\frac{\partial x_j}{\partial \chi_a} - \delta_{ij}, \tag{2}$$

where x_i and χ_a are the Cartesian coordinates of the point at which σ is determined at the times t and t − s, respectively; and δ_{ij} represents the components of the unit tensor.

For strain processes that are smooth in the neighborhood of the point t the tensor function E can be formally expanded in a Taylor series in the neighborhood of the point s = 0:

$$E(t-s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} s^n}{n!} B_n, \tag{3}$$

where

$$B_n = (-1)^{n+1}\left[\frac{d^n E(t-s)}{ds^n}\right]_{s=0}.$$

The elements of the tensor $B_n = (B_{ij}^{(n)})$ have the form

$$B_{ij}^{(1)} = \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i},$$
$$B_{ij}^{(n+1)} = \frac{\partial}{\partial t} B_{ij}^{(n)} + v_k \frac{\partial B_{ij}^{(n)}}{\partial x_k} - \frac{\partial v_i}{\partial x_m} B_{mj}^{(n)} - \frac{\partial v_j}{\partial x_m} B_{im}^{(n)}. \tag{4}$$

Here the v_i are the components of the particle velocity vector of the medium at time t.

It is clear that a certain region $0 \le s \le s_1$ always exists in which the series (3) converges. For real materials the functional F has the property of diminishing memory. This means that the strains of the

Institute of Mechanics, Academy of Sciences of the Ukrainian SSR, Kiev. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 18, No. 6, pp. 1053-1060, June, 1970. Original article submitted September 29, 1969.

medium in the recent past exert a greater influence on the present stress magnitude than do strains in the distant past.

Consequently, for many materials and in many situations, in particular the cases of thermoplastic polymers at high temperatures and dilute polymer solutions, the stress tensor in the medium is determined mainly by the behavior of the function E(t − s) in the neighborhood of s = 0, i.e., by the first terms of the series (3).

Substituting (3) into the functional F, we arrive at the expression

$$\sigma = -pI + Q[B_1, B_2, B_3, \ldots], \tag{5}$$

in which Q is a symmetric tensor function of B_i (i = 1, 2, 3, . . .).

If we assume that the function Q is a polynomial in the tensors B_i, we can invoke the theorem on the transformation of matrix polynomials [3] to reduce Q to canonical form for a very broad class of steady flows of the fluids in question.

It turns out in the final analysis that for many flows the equation of state (1) is equivalent to the expression

$$\sigma = -pI + \varphi_1 B_1 + \varphi_2 B_1^2 + \varphi_3 B_2, \tag{6}$$

in which the φ_i are functions of invariants of the tensors B_1 and B_2.

The representation (6) of relation (1) is exact for certain steady flows, such as simple shear [2], Couette, Poiseuille, and general rectilinear [4] flows, flows generated by the rotation of bodies of revolution [5], and helicoidal flow [6]. For some flows, such as rectilinear flow in an arbitrary cylindrical tube, the transition from expression (1) to relation (6) incurs a very slight error [7]. We shall take for granted, therefore, that the equation of state of the given medium is described by relation (6).

It is easily verified that the equation of state (6) admits the existence of a potential function Φ depending on invariants of the tensors B_1 and B_2, such that

$$\sigma_{ij} + p\delta_{ij} = \tau_{ij} = \frac{\partial \Phi}{\partial B_{ij}^{(1)}}. \tag{7}$$

Hence it follows that the functions φ_i and Φ can be represented in the form

$$\varphi_1 = \frac{df_1(I_2)}{dI_2} + \frac{2}{3} I_3 \frac{df_2(I_2)}{dI_2} + 2 I_7 \frac{df_3(I_2)}{dI_2}; \quad \varphi_2 = f_2(I_2); \; \varphi_3 = f_3(I_2); \tag{8}$$

$$\Phi = \frac{1}{2} f_1(I_2) + \frac{1}{3} I_3 f_2(I_2) + I_7 f_3(I_2), \tag{9}$$

where

$$I_2 = B_{ik}^{(1)} B_{ki}^{(1)}; \; I_3 = B_{ij}^{(1)} B_{jk}^{(1)} B_{ki}^{(1)}; \; I_7 = B_{ij}^{(1)} B_{ji}^{(2)}.$$

The functions $df_1(I_2)/dI_2$, $f_2(I_2)$, and $f_3(I_2)$, can be determined experimentally. The dependence $df_1(I_2)/dI_2$ is found from viscosimetric tests. The function $f_2(I_2)$ can be determined from the secondary difference between the normal stresses in flows close to simple shear flow [8], and methods for the determination of $f_3(I_2)$ are described in [5].

§ 2. Variational Problems Associated with the Motion of Nonlinear Viscoelastic Fluids

The solutions of the motion problems for the investigated media reduce to the integration of equations in the vector velocity function $\bar{v}$, which are obtained by substitution of the equation of state (6) into the set of stress equations of motion; for stationary processes, neglecting inertial terms, these equations have the form

$$\frac{\partial \sigma_{ij}}{\partial x_j} + F_i = 0. \tag{10}$$

Here the F_i are the components of the volumetric force vector.

In view of the incompressibility of the medium the vector function $\bar{v}$ must also satisfy the equation

$$\operatorname{div} \bar{v} = 0. \tag{11}$$

The boundary conditions have the form

$$\bar{v} = 0 \text{ on } S_1, \tag{12}$$

$$\sigma_{ij}(\bar{v}) n_j = N_i \text{ on } S. \tag{13}$$

Here $S_1 \subset S_0$; S_0 is the boundary of the given flow domain Ω; $S = S_0 \setminus S_1$; $\sigma_{ij}(\bar{v})$ is a function describing the components of the stress tensor for the velocity vector function $\bar{v}$, i.e., $\sigma_{ij}(\bar{v})$ can be regarded as the value of a certain nonlinear operator on the vector function $\bar{v}$ [this operator is determined by Eqs. (6) and (4)]; n_j is the cosine of the angle between the normal to the surface and the j-th coordinate axis; and the N_i are functions specified on S.

Consequently, on the motionless surface S_1 bounding the given flow domain the velocity of the fluid is set equal to zero, and on the rest of the surface (on S) it is given by the distribution of the components N_i of the surface forces $\bar{N}$. We assume that $N_i \in C(S)$ and $F_i \in C(\Omega)$.

We now consider the minimization problem for the functional

$$\Psi(\bar{v}) = \int_\Omega \left[\Phi - \left\{ \frac{\partial \Phi}{\partial B_{ij}^{(2)}} \right\} B_{ij}^{(2)} \right] d\Omega - 2 \int_S N_i v_i \, dS - 2 \int_\Omega F_i v_i \, d\Omega. \tag{14}$$

Here Φ is given by relation (9). The braces in (14) signify that the term $\partial\Phi/\partial B_{ij}^{(2)}$ is determined by solving the given problem and is not varied.

More precisely, we state the following problem: in the class B of all admissible vector functions, i.e., among solenoidal vector functions continuous in Ω together with their partial derivatives through third order and zero-valued on S_1, find the vector function that minimizes the functional (14). Clearly, B is a linear manifold.

Let $\bar{v}$ minimize the functional (14); we analyze the value of the functional for the vector function $\bar{v} + \alpha\bar{h}$, where α is a number and $\bar{h}$ is any vector function from the admissible class.

Now

$$\Psi(\bar{v} + \alpha\bar{h}) \geqslant \Psi(\bar{v}) \tag{15}$$

for all α.

The functions $f_i(I_2)$ $(i = 1, 2, 3)$ are assumed to be continuous on the semiaxis $[0, \infty)$ together with their first- and second-order derivatives. Then the derivative $(d/d\alpha)\Psi(\bar{v} + \alpha\bar{h})$ exists. For given vector functions $\bar{v}$ and $\bar{h}$ the expression $\Psi(\bar{v} + \alpha\bar{h})$ may be regarded as a function of α. Inasmuch as this function is minimized for $\alpha = 0$, at the latter point its derivative with respect to α is zero, i.e.,

$$\left[\frac{d}{d\alpha} \Psi(\bar{v} + \alpha\bar{h}) \right]_{\alpha=0} = \int_\Omega \tau_{ij}(\bar{v}) B_{ij}^{(1)}(\bar{h}) \, d\Omega - 2 \int_S N_i h_i \, dS - 2 \int_\Omega F_i h_i \, d\Omega = 0. \tag{16}$$

Here $B_{ij}^{(1)}(\bar{h})$ represents the components of the strain rate vector for the vector function $\bar{h}$; and $\tau_{ij}(\bar{v})$ represents the components of the stress deviator for the vector $\bar{v}$.

Recognizing that $\operatorname{div} \bar{h} = 0$ and taking relation (4) and the Ostrogradskii equation into account, we transform the first term in (16):

$$\int_\Omega \tau_{ij}(\bar{v}) B_{ij}^{(1)}(\bar{h}) \, d\Omega = \int_\Omega \sigma_{ij}(\bar{v}) B_{ij}^{(1)}(\bar{h}) \, d\Omega = 2 \int_\Omega \frac{\partial}{\partial x_j} (\sigma_{ij}(\bar{v}) h_i) \, d\Omega$$
$$- 2 \int_\Omega \frac{\partial \sigma_{ij}(\bar{v})}{\partial x_j} h_i \, d\Omega = 2 \int_S h_i \sigma_{ij}(\bar{v}) n_j \, dS - 2 \int_\Omega \frac{\partial \sigma_{ij}(\bar{v})}{\partial x_j} h_i \, d\Omega. \tag{17}$$

Substituting the resulting expression into relation (16), we find

$$\int_{\Omega}\left(\frac{\partial\sigma_{ij}(\bar{v})}{\partial x_j}+F_i\right)h_i\,d\Omega-\int_{S}(\sigma_{ij}(\bar{v})n_j-N_i)\,h_i\,dS=0. \tag{18}$$

Since the latter equation must be valid for any vector function in class B, we can, in particular, choose $\bar{h}$ equal to zero on S, whereupon we have

$$\int_{\Omega}\left(\frac{\partial\sigma_{ij}(\bar{v})}{\partial x_j}+F_i\right)h_i\,d\Omega=0. \tag{19}$$

Relation (19) holds for any vector function $\bar{h}\in$ B that vanishes on S. We now show that this situation is possible only if $\bar{v}$ satisfies the system of differential equations

$$\frac{\partial\sigma_{ij}(\bar{v})}{\partial x_j}+F_i=0 \qquad (i=1,\ 2,\ 3). \tag{20}$$

We proceed by assuming the opposite, denoting by $\bar{R}$ the vector function defined by the components $R_i=\frac{\partial\sigma_{ij}(\bar{v})}{\partial x_j}+F_i$. Then, since $\bar{v}\in$ B, the function $\bar{R}$ will assume nonzero values, if not over the entire domain Ω, then at least on some set $\Omega_1\subset\Omega$ having nonzero measure. If we construct a vector function $\bar{h}\in$ B, zero-valued on S_0, such that the direction $\bar{R}$ coincides with the direction of $\bar{h}$ at every point of Ω in which $\bar{R}$ is nonvanishing, we have

$$\int_{\Omega}\bar{R}\bar{h}\,d\Omega=\int_{\Omega}\left(\frac{\partial\sigma_{ij}(\bar{v})}{\partial x_j}+F_i\right)h_i\,d\Omega>0, \tag{21}$$

which contradicts Eq. (19). Consequently, all that remains is to prove that such a vector function $\bar{h}$ does exist. The vector function we seek, $\bar{h}=(h_1,\ h_2,\ h_3)$, can be constructed from the relations

$$\frac{h_1}{h_2}=f_1(x_1,\ x_2,\ x_3);\qquad \frac{h_3}{h_2}=f_2(x_1,\ x_2,\ x_3), \tag{22}$$

$$\frac{\partial h_1}{\partial x_1}+\frac{\partial h_2}{\partial x_2}+\frac{\partial h_3}{\partial x_3}=0, \tag{23}$$

$$\bar{h}=0 \text{ on } S_0. \tag{24}$$

Here $f_1(x_1,\ x_2,\ x_3)$ and $f_2(x_1,\ x_2,\ x_3)$ are certain triples of continuously differentiable functions defined at every point $\bar{x}=(x_1,\ x_2,\ x_3)\in\Omega$ by the direction of the vector function $\bar{R}$.

Once the values of h_1 and h_3 have been determined from Eqs. (22) and substituted into (23), we can verify that problem (22)-(24) is equivalent to the following:

$$\frac{\partial}{\partial x_1}(h_2f_1)+\frac{\partial h_2}{\partial x_2}+\frac{\partial}{\partial x_3}(h_2f_2)=0, \tag{25}$$

$$h_2=0 \quad \text{on } S_0. \tag{26}$$

If the boundary S_0 of the domain Ω is sufficiently smooth, it can be specified in the parametric form

$$x_k=x_k(t_1,\ t_2) \quad (k=1,\ 2,\ 3), \tag{27}$$

so as to make the right-hand sides of these equations continuous and have continuous first-order derivatives in a certain domain D of the two-dimensional space $(t_1,\ t_2)$.

Then our problem reduces to the determination of an integral surface for (25) containing the following two-dimensional manifold:

$$x_k=x_k(t_1,\ t_2),\quad h_2=h_2\left(x_k(t_1,\ t_2)\right)=0 \quad (k=1,\ 2,\ 3). \tag{28}$$

If the determinant

$$\Delta = \begin{vmatrix} f_1 & 1 & f_2 \\ \frac{\partial x_1}{\partial t_1} & \frac{\partial x_2}{\partial t_1} & \frac{\partial x_3}{\partial t_1} \\ \frac{\partial x_1}{\partial t_2} & \frac{\partial x_2}{\partial t_2} & \frac{\partial x_3}{\partial t_2} \end{vmatrix} \quad (29)$$

has a nonzero value on the manifold (28), then a solution exists for the problem (25), (28) [9].

Therefore, given fairly general assumptions regarding the boundary of the analyzed domain, the vector function $\bar{v}$ minimizing the functional (14) in class B satisfies the system of differential equations (20).

Now from relations (18) we have

$$\int_S \left(\sigma_{ij}(\bar{v})\, n_j - N_i\right) h_i dS = 0. \quad (30)$$

Hence it follows that the vector function $\bar{v}$ satisfies the boundary conditions (13), as otherwise, constructing a vector function $\bar{h} \in B$ such that the direction of $\bar{h}$ coincides at every point of S with the direction of the vector defined by the components $U_i = \sigma_{ij}(\bar{v})\, n_j - N_i$, we arrive by analogy with the foregoing discussion at an inequality that contradicts Eq. (30).

We can consolidate the results to now in the form of a theorem.

THEOREM. Under the above-stated conditions, if a vector function $\bar{v} \in B$ minimizes the functional (14) in class B, then $\bar{v}$ satisfies the system of equilibrium differential equations in the velocity components (20) and the boundary condition (13).

Thus, the solution of the motion problem for a nonlinear viscoelastic fluid, i.e., the solution of the system of equations (20) and (11) subject to the boundary conditions (12) and (13), reduces to searching for an extremal point of the functional (14) in class B. It follows from the theorem that the boundary conditions (13) are natural, so that there need not be special concern for the satisfaction of condition (13) in solving the variational problem by the Ritz method.

Note that the desired vector function $\bar{v}$ is determined as the limit in a certain metric of a minimizing sequence $\{\bar{v}_n\}$ formed from elements of class B [provided that this sequence converges, i.e., $\bar{v}_n \to \bar{v}$, and that $\lim_{n\to\infty} \Psi(v_n) = \Psi(v)$] [10, 11], and it can turn out that this limit has poorer differential properties than a vector function of class B. In this case we call $\bar{v}$ a generalized solution of the problem (20), (11), (12), (13).

§ 3. Rectilinear Motion of a Nonlinear Viscoelastic Fluid for a Nonuniform Temperature Distribution

We now investigate a nonisothermal flow of a nonlinear viscoelastic fluid in an arbitrary cylindrical duct. We introduce a motionless Cartesian coordinate system (x_1, x_2, x_3) in such a way that the x_3 axis is parallel to the generatrix of the cylinder. We assume that the flow process is steady and is directed along the axis of the duct. We assume, further, that the temperature T in the medium depends only on the coordinates x_1 and x_2, i.e., that

$$T = T(x_1, x_2), \quad (31)$$

where the function $T(x_1, x_2)$ is considered to be known and continuously differentiable in the domain of the cylinder cross section, Ω. We also say that the components of the velocity vector are determined by the expressions

$$v_3 = v(x_1, x_2); \quad v_1 = v_2 = 0. \quad (32)$$

In the general case of the strain of a nonlinear viscoelastic fluid under conditions of a nonuniform temperature distribution, the stress tensor in the medium is determined, correct to the hydrostatic pressure, by the strain and temperature history [12]. By virtue of relations (31) and (32) the temperature of points in the fluid remains constant at times preceding the present. Then the stress tensor in the fluid is determined at every point of the domain of the cylinder cross section, Ω, by relation (1); only the functional F is determined by the temperature of the medium at a point, i.e., F is an abstract function of the numerical argument T, and we denote it by F_T. Now, by analogy with relation (1), we have the following expression for the stress tensor:

$$\sigma = -pI + \overset{\infty}{\underset{s=0}{F_T}} \left[E(t-s)\right]. \tag{33}$$

As in [4], we can verify that the stresses at every point are determined by the expression

$$\sigma = -pI + \varphi_{1,T}(I_2) B_1 + \varphi_{2,T}(I_2) B_1^2 + \varphi_{3,T}(I_2) B_2, \tag{34}$$

in which the $\varphi_{i,T}(I_2)$ are functions of the invariant I_2 and at every point of Ω are determined by the value of T corresponding to the function $\varphi_{i,T}(I_2)$. Thus, the functions $\varphi_{i,T}(I_2)$ can be treated as abstract functions of the argument T, i.e., there is associated with every value of T a numerical function of the argument I_2. The functions $\varphi_{i,T}(I_2)$ determine numerical functions of the arguments I_2 and T, namely $\varphi_i(I_2, T)$, in such a way that $\varphi_{i,q}(\xi) = \varphi_i(\xi, q)$ for any values of ξ and q from the half-strip Λ ($0 \le \xi < \infty$; $a \le q \le b$), where

$$a = \min_{(x_1,x_2)\in\overline{\Omega}} T(x_1, x_2); \quad b = \max_{(x_1,x_2)\in\overline{\Omega}} T(x_1, x_2). \tag{35}$$

The functions $\varphi_i(I_2, T)$ characterize the properties of the material in strain at a constant temperature T. In the given problem, therefore, the equation of state of the fluid reduces to the form

$$\sigma = -pI + \varphi_1(I_2, T) B_1 + \varphi_2(I_2, T) B_1^2 + \varphi_3(I_2, T) B_2. \tag{36}$$

Now from the equations of motion we deduce

$$\frac{\partial p}{\partial x_3} = \frac{\partial}{\partial x_1}\left[\varphi_1(I_2, T(x_1, x_2)) \frac{\partial v}{\partial x_1}\right] + \frac{\partial}{\partial x_2}\left[\varphi_1(I_2, T(x_1, x_2)) \frac{\partial v}{\partial x_2}\right], \tag{37}$$

$$p = \gamma(x_1, x_2) + \lambda x_3, \tag{38}$$

where $\gamma(x_1, x_2)$ is a certain function of x_1 and x_2, $\lambda = \partial p/\partial x_3 = \text{const}$, and

$$I_2 = 2\left[\left(\frac{\partial v}{\partial x_1}\right)^2 + \left(\frac{\partial v}{\partial x_2}\right)^2\right]. \tag{39}$$

The boundary condition provides contiguity between the medium and the surface of the cylinder, i.e.,

$$v = 0 \quad \text{on } S_0. \tag{40}$$

The function $\varphi_3(I_2, T)$ generates stresses that dilate (compress) the fluid and, as apparent from (37), does not affect the velocity distribution in the medium for the analyzed rectilinear flows. The stresses generated by the function $\varphi_2(I_2, T)$ must be equalized by the volumetric forces. In the absence of the latter it is required to set $\varphi_2(I_2, T) = 0$ in order for rectilinear motion of the type (32) to be possible.

Expression (37) has the following implication: in the flows described by relations (31) and (32) a nonlinear viscoelastic fluid in which the stresses are determined by the strain and temperature histories has the same velocity distribution as a nonlinear viscous fluid whose viscosity depends on the invariant I_2 and the temperature T.

As shown in [13], given fairly general assumptions, problem (37), (40), is equivalent to the problem of minimizing the functional

$$\Psi(v) = \frac{1}{4}\int\limits_{\Omega} dx_1 dx_2 \int\limits_0^{I_2} \varphi_1(I_2, T(x_1, x_2))\, dI_2 + \iint\limits_{\Omega} \frac{\partial p}{\partial x_3} v\, dx_1 dx_2,$$

$$v/_{S_0} = 0.$$

Existence and uniqueness conditions for the generalized solution of the investigated problem are also established in the above-cited paper.

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